Higman's Theorem and the Multiset Order

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The following is a proof that the multiset extension of $(\mathbb{N}, \leq)$ is well-founded. The proof uses a generalization of Higman's theorem (1952), which originally considered string embeddings. I closely follow the development (originally attributed to Nash-Williams) in Jean Gallier's article, *What's So Special About Kruskal's Theorem and the Ordinal Γ_0?*.

Let $(A, \preceq)$ be a quasi-order. Its *multiset extension* $(\mathcal{M}(A), \ll)$ is defined by $X \ll Y$ iff $X = (Y - A) \cup B$ for some $A, B \in \mathcal{M}(A)$ with $\emptyset \neq A \subseteq M$, for all $b \in B$ there exists $a \in A$ with $b \preceq a$, and $a \preceq b$ for at most $|A|$ of the elements $b \in B$. $(\mathcal{M}(A), \ll)$ is also a quasi-order.

For an infinite sequence $(a_i)_{i\geq 1}$, a strictly monotonic function $f : \mathbb{N}^+ \to \mathbb{N}^+$ defines the subsequence $(a_{f(i)})_{i\geq 1}$. An infinite sequence $(a_i)_{i\geq 1}$ over $(A, \preceq)$ is *good* if there exist positive integers $i < j$ such that $a_i \preceq a_j$, and *bad* otherwise. If all its infinite sequences are good, then $(A, \preceq)$ is a *well-quasi-order* (WQO).

Lemma 1. *$(A, \preceq)$ is a WQO iff there exists a subsequence $(a_{f(i)})_{i\geq 1}$ such that, for all positive integers i, $a_{f(i)} \preceq a_{f(i+1)}$.*

Proof. (Non-trivial direction.) Assume a is an infinite sequence over A. Call $i > 0$ a *terminal index* in a if there is no $j > i$ such that $a_i \preceq a_j$. There are only finitely many terminal indexes in a. Otherwise, the subsequence of terminal indexes would be bad, contradicting the assumption that $(A, \preceq)$ is a WQO. So let N be the last terminal index. A suitable subsequence $(a_{f(i)})_{i\geq 1}$ is defined such that $f(1) = N + 1$, and $f(i+1)$ is the least index such that $a_{f(i)} \preceq a_{f(i+1)}$, which exists because all terminal indexes are less than $f(1)$.

Theorem 1 (Higman). *If $(A, \preceq)$ is a WQO, then $(\mathcal{M}(A), \ll)$ is also a WQO.*

Proof. If not, then there exists a bad sequence on $\mathcal{M}(A)$. Define a *minimal bad sequence* t on $\mathcal{M}(A)$ such that t_1 is a multiset of minimal size that starts a bad sequence, and t_{n+1} is a multiset of minimal size that is the $n + 1$st element of a bad sequence whose first n elements are $(t_i)_{1\leq i\leq n}$.

Because t is bad and $\emptyset \ll x$ for any $x \in \mathcal{M}(A)$, for all $i > 0$, $|t_i| \geq 1$. Let a_i be a maximal element of t_i. Then $t_i = \{a_i\} \uplus s_i$ for some $s_i \in \mathcal{M}(A)$. By definition of the multiset extension, $s_i \ll t_i$ for all $i > 0$.

By Lem. 1, there is a subsequence $a' = (a_{f(i)})_{i\geq 1}$ of a such that $a_{f(i)} \preceq a_{f(i+1)}$ for all $i > 0$. The subsequence $s' = (s_{f(i)})_{i\geq 1}$ of s is good. If not, and $f(1) = 1$, then the sequence $(s_{f(i)})_{i\geq 1}$ is bad and has $|s_1| < |t_1|$, contradicting minimality of t. Otherwise $f(1) > 1$, and the sequence $\langle t_1, \ldots, t_{f(1)-1}, s_{f(1)}, s_{f(2)}, \ldots\rangle$ is also bad, because if not, for some $i < f(1)$ and $j > 0$, $t_i \ll s_{f(j)} \ll t_{f(j)}$, contradicting the assumption that t is bad. But $|s_{f(1)}| < |t_{f(1)}|$, contradicting minimality of t. So s' is good.

Since s' is good, there exists positive integers i, j such that $f(i) < f(j)$ and $s_{f(i)} \ll s_{f(j)}$. But by definition of a', $a_{f(i)} \preceq a_{f(j)}$, and so

$$t_{f(i)} = \{a_{f(i)}\} \uplus s_{f(i)} \ll \{a_{f(j)}\} \uplus s_{f(j)} = t_{f(j)},$$

which contradicts assumption that t is bad.

Theorem 2. *The multiset extension of* $(\mathbb{N}, \leq)$, $(\mathcal{M}(\mathbb{N}), \ll)$, *is well-founded.*

Proof. $(\mathbb{N}, \leq)$ is a WQO because $\leq$ is total and well-founded. By Thm. 1, $\mathcal{M}(\mathbb{N}, \ll)$ is a WQO. All the infinite sequences of a WQO are good, and good sequences are not infinitely decreasing. Hence, $\mathcal{M}(\mathbb{N}, \ll)$ is well-founded.